COMSOL CONFERENCE
2017 BOSTON

Simulation Of A Vortex Ring: Dealing With The Unbounded, Doubly Connected Domain

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COMSOL Conference, Boston, October 5, 2017

Objective

Develop a COMSOL simulation such that:

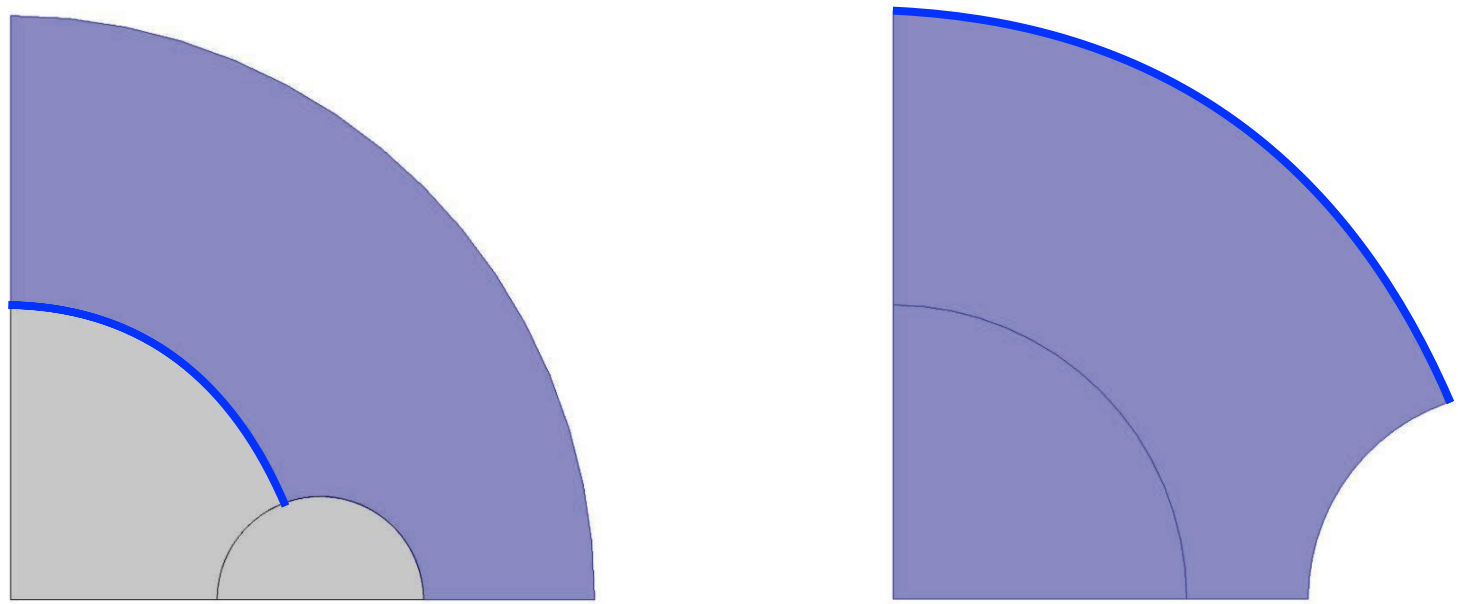
1 The region, $\mathcal{R}$, occupied by the fluid is unbounded;

2 There is a subregion, $\mathcal{R}^c$, in which the motion is rotational (*i.e.* the velocity field, $\mathbf{u}$ satisfies $\operatorname{curl}(\mathbf{u}) \neq \mathbf{0}$);

3 There is a *doubley connected* subregion, $\mathcal{R}^c \backslash \mathcal{R}$, in which the motion is irrotational, *i.e.* $\operatorname{curl}(\mathbf{u}) = \mathbf{0}$.

Domain mapping by Kelvin Inversion

Physical (R, θ, ϕ) Proxy (q, ϑ, φ)



Radial		Colatitudinal		Azimuthal
$R/a = (q/a)^{-1}$	,	$\theta = \vartheta$	,	$\phi = \varphi$,
$\hat{\mathbf{e}}_R = \hat{\mathbf{u}}_q$	,	$\hat{\mathbf{e}}_\theta = \hat{\mathbf{u}}_\vartheta$	,	$\hat{\mathbf{e}}_\phi = \hat{\mathbf{u}}_\varphi$,
$R\frac{\partial}{\partial R} = -q\frac{\partial}{\partial q}$	,	$\frac{\partial}{\partial \theta} = \frac{\partial}{\partial \vartheta}$	,	$\frac{\partial}{\partial \phi} = \frac{\partial}{\partial \varphi}$.

Field equations, physical domain, I

Let $\mathcal{R}$, $\mathcal{R}^c \subset \mathcal{R}$, and $\mathcal{R}\backslash\mathcal{R}^c$ denote all of physical space, the region occupied by the vortex core, and the compliment of $\mathcal{R}^c$ in $\mathcal{R}$, respectively. Let $\mathbf{u}$ denote the velocity field. I assume that

$$\operatorname{div}\mathbf{u} = 0 \quad \forall\,\mathbf{R} \in \mathcal{R},$$

$$\operatorname{curl}\mathbf{u} = \begin{cases} Ar\hat{\mathbf{e}}_\phi, & \forall\,\mathbf{R} \in \mathcal{R}^c, \\ \mathbf{0}, & \forall\,\mathbf{R} \in \mathcal{R}\backslash\mathcal{R}^c, \end{cases}$$

in which (r, ϕ, z) are cylindrical coordinates, $\{\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\phi, \hat{\mathbf{k}}\}$ is the corresponding system of unit vectors, and A is a constant.

Transformation to a second order system

We have $\operatorname{curl}\mathbf{u} = \omega_\phi \hat{\mathbf{e}}_\phi$, in which $\omega_\phi := \partial u_r/\partial z - \partial u_z/\partial r$. Let

$$\mathbf{F} := \begin{cases} -(\omega_\phi - Ar)\hat{\mathbf{e}}_r + (2\operatorname{div}\mathbf{u})\hat{\mathbf{k}} & \forall\, \mathbf{R} \in \mathcal{R}^c, \\ -\omega_\phi \hat{\mathbf{e}}_r + (2\mathrm{div}\,\mathbf{u})\hat{\mathbf{k}} & \forall\, \mathbf{R} \in \mathcal{R}^c \backslash \mathcal{R}. \end{cases}$$

The field equations therefore require that $\mathbf{F}$ vanish identically. One may ensure that it does by requiring that

$$\operatorname{curl}\mathbf{F} = \mathbf{0} \quad \text{and} \quad \operatorname{div}\mathbf{F} = 0$$

subject to $\mathbf{F} \bullet \hat{\mathbf{n}} = 0$ on $\partial\mathcal{R}$.

Field equations, physical domain, II

For $\mathbf{R} \in \mathcal{R}$ the equations $\operatorname{curl} \mathbf{F} = \mathbf{0}$ and $\operatorname{div} \mathbf{F} = 0$ are equivalent to

$$\begin{pmatrix} \partial/\partial r & \partial/\partial z \end{pmatrix} \begin{pmatrix} 2 \operatorname{div} \mathbf{u} \\ (\omega_\phi - Ar) \end{pmatrix} = 0$$

and

$$\begin{pmatrix} \partial/\partial r & \partial/\partial z \end{pmatrix} \begin{pmatrix} -r(\omega_\phi - Ar) \\ r\, 2 \operatorname{div} \mathbf{u} \end{pmatrix} = 0\,,$$

respectively, which are suited to programming in COMSOL's General Form PDE physics interface (with the default Zero Flux boundary condition).

Field equations, proxy domain, I

For $\mathbf{q} \in \mathcal{Q}$ the equations $\operatorname{curl} \mathbf{F} = \mathbf{0}$ and $\operatorname{div} \mathbf{F} = 0$ are equivalent to

$$\begin{pmatrix} \partial/\partial\varpi & \partial/\partial\zeta \end{pmatrix} \begin{pmatrix} \varpi\, F_\varpi \\ \varpi\, F_\zeta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} ,$$

$$\begin{pmatrix} \partial/\partial\varpi & \partial/\partial\zeta \end{pmatrix} \begin{pmatrix} F_\zeta \\ -F_\varpi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} ,$$

in which

$$\varpi = q \sin\vartheta \quad , \quad \zeta = q \cos\vartheta$$

are cyindrical coordinates in the proxy domain,

Field equations, proxy domain, II

$$F_\varpi = 2(u_\varpi/\varpi)a^{-2}S$$
$$+\,a^{-2}[2u_{\zeta,\zeta}S - (u_{\varpi,\zeta} + u_{\zeta,\varpi})C]$$
$$-\,2a^{-2}[S^2 u_{\varpi,\varpi} + SCu_{\varpi,\zeta} + CSu_{\zeta,\varpi} + C^2 u_{\zeta,\zeta}]S \ ,$$

$$F_\zeta = 2(u_\varpi/\varpi)a^{-2}C$$
$$-\,a^{-2}[(u_{\varpi,\zeta} + u_{\zeta,\varpi})S - 2u_{\varpi,\varpi}C]$$
$$-\,2a^{-2}[S^2 u_{\varpi,\varpi} + SCu_{\varpi,\zeta} + CSu_{\zeta,\varpi} + C^2 u_{\zeta,\zeta}]C \ ,$$

in which the commas denote partial differentiation, $\mathbf{u} = u_\varpi \hat{\mathbf{u}}_\varpi + u_\zeta \hat{\mathbf{u}}_\zeta$, and

$$C := \zeta/q \quad , \quad S := \varpi/q \quad , \quad q = (\varphi^2 + \zeta^2)^{1/2} \ .$$

Flux boundary condition for proxy domain, and continuity by proxy across a portal

As in the physical domain COMSOL's default Zero Flux boundary condition is appropriate in the proxy domain and suffices to ensure that $\operatorname{div}\mathbf{u} = 0$ and $\operatorname{curl}\mathbf{u} = \mathbf{0}$ there. Here I distinguish between a *boundary* and a *portal*. The latter is an internal boundary across which one specifies conditions that ensure *continuity by proxy* for both the velocity components and the fluxes.

Boundary conditions for normal and tangential velocity, I

Let $\hat{\mathbf{n}}$ denote the outward unit normal vector on any boundary and let $\hat{\mathbf{t}} = \hat{\mathbf{e}}_\phi \times \hat{\mathbf{n}}$ be the corresponding unit tangent.

In the upper half of $\mathcal{R}^c$ the velocity $\mathbf{u}$ is unique provided one specifies values of $\mathbf{u} \bullet \hat{\mathbf{n}}$ on part of the boundary (in this case the core boundary) and $\mathbf{u} \bullet \hat{\mathbf{t}}$ on the rest (in this case the equitorial edge).

Boundary conditions for normal and tangential velocity, II

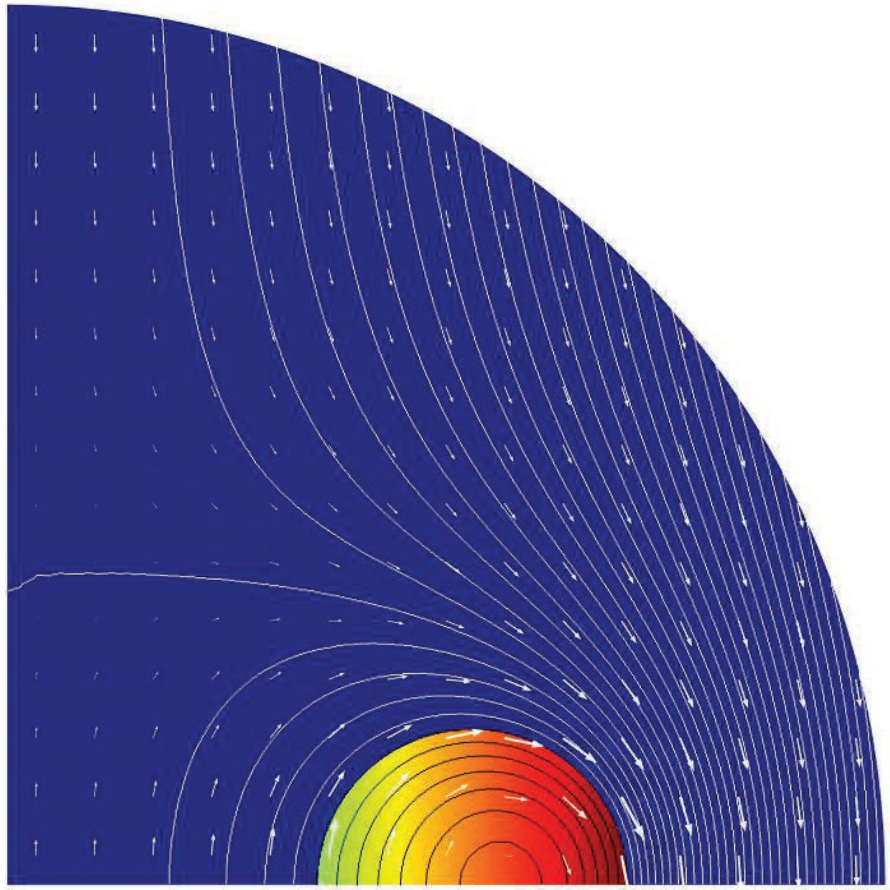
In the upper half of $\mathcal{R}^c \backslash \mathcal{R}$ (including the proxy domain) the velocity $\mathbf{u}$ is *not* unique even if one specifies values of $\mathbf{u} \bullet \hat{\mathbf{n}}$ on two disjoint parts of the boundary (*viz.* the centerline and the core boundary) and $\mathbf{u} \bullet \hat{\mathbf{t}}$ on the rest (*viz.* the two disjoint equitorial edges). A supplementary condition, namely specification of the value of Γ in the integral constraint $\int_{\mathcal{C}} \mathbf{u} \bullet \hat{\mathbf{t}}\, ds = \Gamma/2$, in which $\mathcal{C}$ is the upper core boundary, *does* ensure uniqueness.

On propagation speed of the ring, I

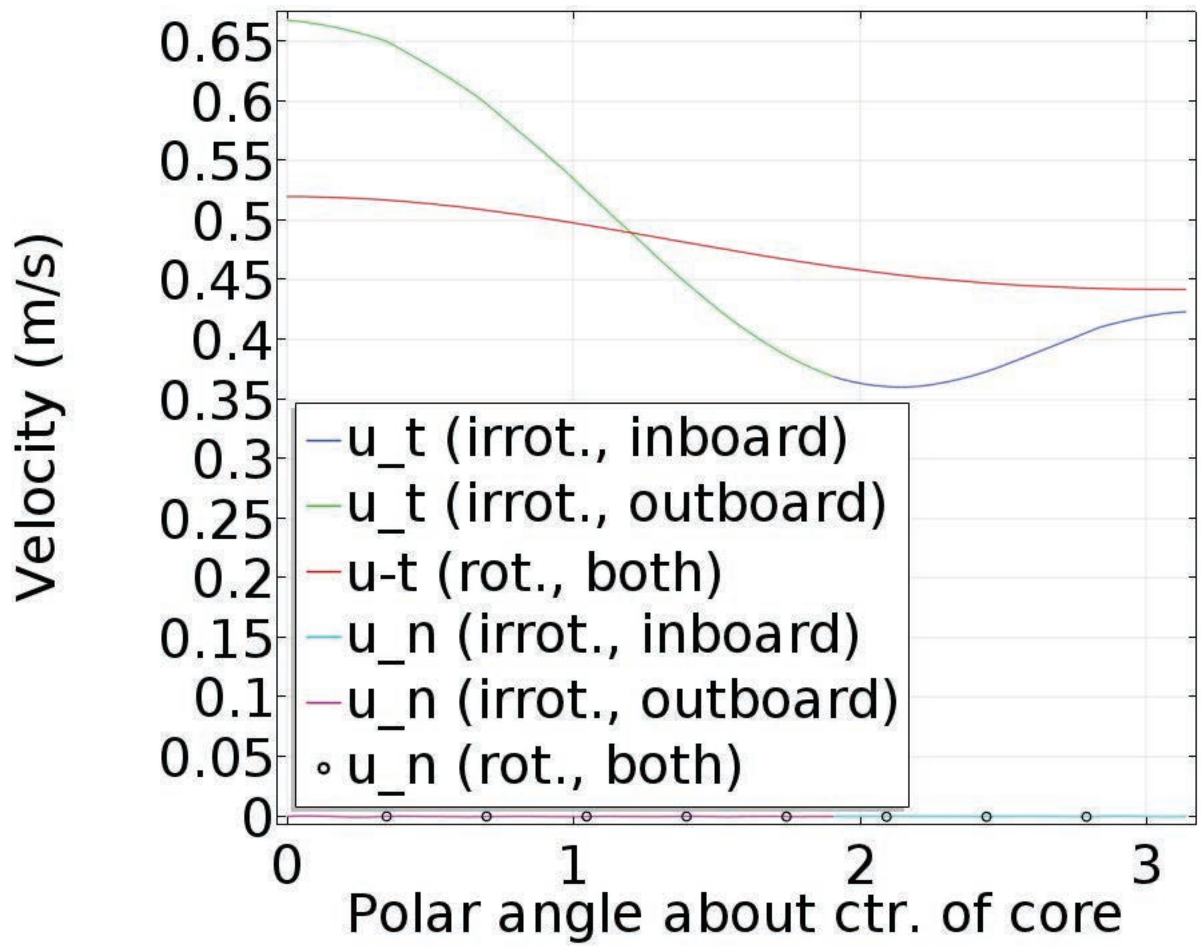
One may calcuate the propagation speed of the ring in two ways, depending on the choice of reference frame. In one, the observer moves with the ring and the remote fluid is in uniform motion at a unknown speed, W_∞. In COMSOL one may introduce a physics interface of the form Global ODEs and DAEs with dependent variable W_∞ and constraint expression $\int_{\mathcal{C}} \mathbf{u} \bullet \hat{\mathbf{t}}\, ds - \Gamma/2$. For this choice of reference frame $\mathbf{u} \bullet \hat{\mathbf{t}} = 0$ on both equitorial edges and $\mathbf{u} \bullet \hat{\mathbf{n}} = 0$ on the centerline and the core boundary.

On propagation speed of the ring, II

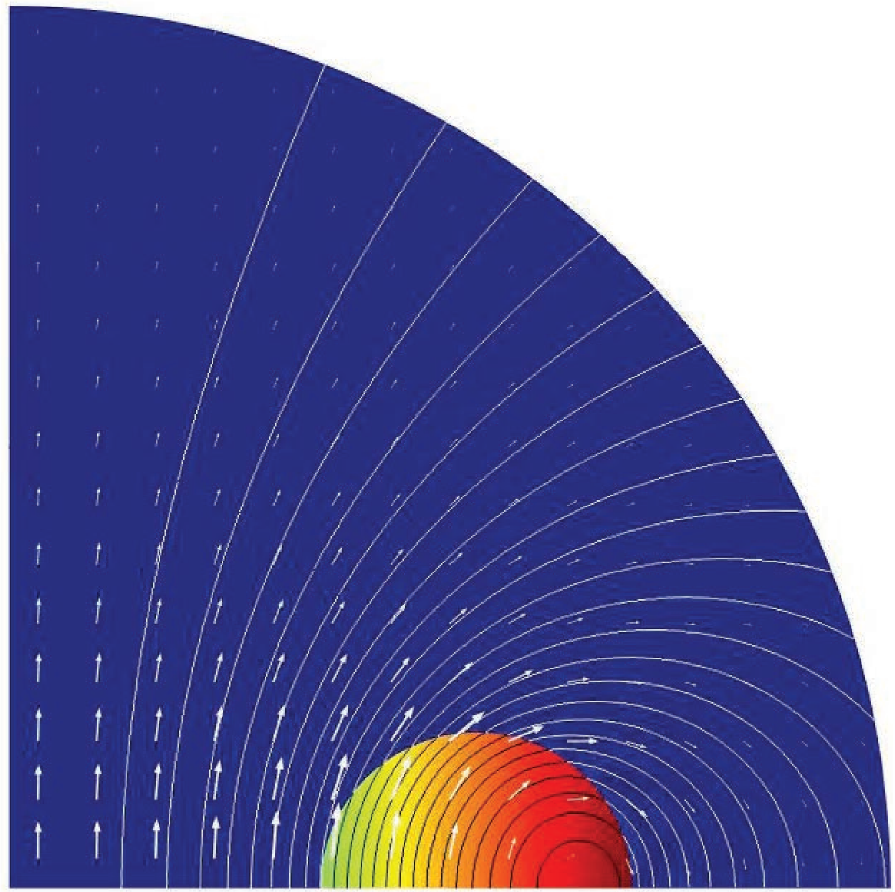
In another choice of reference frame the observer is at rest relative to the remote fluid and the ring rises uniformly at a unknown speed, W_{rise}. This time in the Global ODEs and DAEs physics interface the dependent variable is W_{rise} and again the constraint expression is $\int_{\mathcal{C}} \mathbf{u} \bullet \hat{\mathbf{t}}\, ds - \Gamma/2$. For this choice of reference frame $\mathbf{u} \bullet \hat{\mathbf{t}} = 0$ on both equitorial edges and $\mathbf{u} \bullet \hat{\mathbf{n}} = 0$ on the centerline but now $\mathbf{u} \bullet \hat{\mathbf{n}} = W_{\text{rise}} \hat{\mathbf{k}} \bullet \hat{\mathbf{n}}$ on the core boundary.



Meridional cut through a vortex ring as seen by an observer moving with the ring showing streamlines and velocity vectors. Colors denote azimuthal vorticity.



Tangential (u_t) and normal (u_n) components of velocity over the core boundary on its two sides, corresponding to rotational and irrotational motion.



Meridional cut through a vortex ring as seen by an observer at rest relative to the remote fluid showing streamlines and velocity vectors. Colors denote azimuthal vorticity.

Conclusions (Part 1 of 2)

1. COMSOL enables computation of a solenoidal velocity field in a physical domain $\mathcal{R}^i \cup \mathcal{R}^e$—in which $\mathcal{R}^i$ and $\mathcal{R}^e$ denote a bounded interior and an unbounded exterior regions, respectively—by simultaneous solution for the flows in $\mathcal{R}^i$ and $\mathcal{Q}$, in which $\mathcal{Q}$ is a *bounded proxy* for $\mathcal{R}^e$;
2. COMSOL's General Extrusion model coupling operator enables one to give effect to the change of independent variable (Kelvin Inversion) that maps $\mathcal{R}^e$ to $\mathcal{Q}$;

Conclusions (Part 2 of 2)

3 The simultaneous assumptions that the vortex core has circular cross section and that the azimuthal vorticity in the core is directly proportional to the distance from the centerline enables one to satisfy continuity of normal but not tangential velocity across the core boundary.